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Spin^c-structures and Dirac operators on contact manifolds

Robert Petit

Laboratoire de Mathématiques Jean Leray, UMR 6629 CNRS, Université de Nantes,
2, rue de la Houssinière BP 92208, 44322 Nantes, France

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Abstract

Any contact metric manifold has a Spin^c-structure. Thus, we study on any Spin^c-spinor bundle of a contact metric manifold, Dirac type operators associated to the generalized Tanaka–Webster connection. Bochner–Lichnerowicz type formulas are derived in this setting and vanishing theorems are obtained.

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1. Introduction

In the last years, Spin^c-structures have played an important part in the geometry and the topology of manifolds, and especially in dimension four by means of the Seiberg–Witten theory. Among all the manifolds endowed with a Spin^c-structure, a central part is played by the almost hermitian manifolds. Actually, any almost hermitian manifold has a canonical Spin^c-structure determined by its almost hermitian structure. A contact metric structure on a contact manifold is the data of an almost complex structure on the contact distribution together with a metric compatible both with the almost complex structure and the contact form. A contact manifold endowed with a contact metric structure is refered as a contact metric manifold. Any contact metric manifold has a canonical Spin^c-structure determined by its contact

E-mail address: petit@math.univ-nantes.fr (R. Petit).

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metric structure. On the other hand, recall that a basic tool in the study of contact metric manifolds is the generalized Tanaka–Webster connection [18]. This connection preserves both the contact form and the compatible metric (and also preserves the almost complex structure in the setting of contact metric *CR* manifolds so-called strictly pseudoconvex *CR* manifolds). Therefore, given a Spin^c-structure on a contact metric manifold, then we can define a spinorial connection on the associated spinor bundle by means of the generalized Tanaka–Webster connection together with a connection on the determinant line bundle. A Dirac operator is canonically associated to a such connection. We also define, by restriction to the contact distribution, a sub-Riemannian analogue of this operator called in the following the Kohn–Dirac operator.

One of purposes of this article is to derive some Bochner–Lichnerowicz formulas for such operators. The main motivation being to obtain vanishing theorems for the harmonic and subharmonic spinors (i.e., spinors respectively in the kernel of the Dirac operator and the Kohn–Dirac operator).

The plan of this article is the following. In Section 2, we recall basic facts concerning the contact metric manifolds and the generalized Tanaka–Webster connection. The definitions of curvatures, and specially of the Tanaka–Webster scalar curvature, are given, as also Bianchi identities. In Section 3, we discuss the canonical Spin^c-structure of a contact metric manifold and Dirac type operators associated to the generalized Tanaka–Webster connection. In the particular setting of strictly pseudoconvex *CR* manifolds, we note that the Kohn–Dirac operator is a square root of the Kohn Laplacian. The Section 4 is devoted to obtain Bochner–Lichnerowicz type formulas for the Kohn–Dirac operator (Proposition 4.2). In particular, we derive a Spin^c-version of the Weitzenbock–Tanaka formula in [17]. As a main application, we obtain the following vanishing theorems for the harmonic and subharmonic spinors: *there is no harmonic spinor on a compact spin contact metric manifold if its Tanaka–Webster torsion vanishes and its Tanaka–Webster scalar curvature is nonnegative and positive at some point* (Theorem 4.2).

Any subharmonic spinor ψ on a compact spin strictly pseudoconvex CR manifold with pseudo-Hermitian Ricci tensor nonnegative and pseudo-Hermitian scalar curvature positive at some point can be decomposed as $\psi = \psi_1 + \psi_2$, where ψ_1 and $\overline{\psi}_2$ are holomorphic sections of a square root of the canonical bundle (Corollary 4.2).

In Section 5, we explain the Bochner–Lichnerowicz formula (10) of Proposition 4.2 on a spin contact metric manifold in terms of infinitesimal variations of the Kohn–Dirac operator with respect to some variations of the metric associated to the contact metric structure (Proposition 5.1). To conclude, Nicolaescu considers in [13] Dirac operators on contact metric manifolds associated to contact connections different from the generalized Tanaka–Webster connection. These connections preserve the almost complex structure on the contact distribution even in the almost *CR* case. However, in comparison with the generalized Tanaka–Webster connection, the torsion terms are in general more complicated.

2. Contact metric manifolds

A contact form on a smooth manifold M of dimension $m = 2d + 1$ is a 1-form θ satisfying $\theta \wedge (d\theta)^d \neq 0$ everywhere on M. If θ is a contact form on M, the hyperplan subbundle H of TM given by $H = \operatorname{Ker}\theta$ is called a contact structure. The Reeb field associated to θ is the unique vector field ξ on M satisfying $\theta(\xi) = 1$ and $d\theta(\xi, .) = 0$. By a contact manifold (M, θ) we mean a manifold M endowed with a fixed contact form θ.

Let (M,θ) be a contact manifold, then the pair $(H, d\theta_{\lceil H})$ is a symplectic vector bundle. We recall that an almost complex structure J on a symplectic vector bundle V is compatible with a symplectic form ω if, $\omega(X, JY) = -\omega(JX, Y)$, for any $X, Y \in V$, and, $\omega(X, JX) > 0$, for any $X \in V/\{0\}$. We fix an almost complex structure J on H compatible with $d\theta_{\lceil H}$. Hence, $g_{\theta,H}$ given by $g_{\theta,H}(X,Y) = d\theta(X, JY)$ defines a Hermitian metric on H. We extend J on TM by $J\xi = 0$. This allows to extend $g_{\theta,H}$ to a Riemannian metric g_θ on TM (called the Webster metric) by setting

$$g_\theta(X,Y) = d\theta(X, JY) + \theta(X)\theta(Y).$$

Note that the following relations hold:

$$g_\theta(\xi, X) = \theta(X), \qquad J^2 = -\mathrm{Id} + \theta \otimes \xi, \qquad g_\theta(JX, Y) = d\theta(X,Y), \quad X, Y \in TM.$$

The metric g_θ is said to be associated to θ and we denote by $\mathcal{M}(\theta)$ the set of metrics associated to θ. We call $(\theta, \xi, J, g_\theta)$ a contact metric structure and $(M, \theta, \xi, J, g_\theta)$ a contact metric manifold (cf. Blair [2]).

Let $(M, \theta, \xi, J, g_\theta)$ be a contact metric manifold. Remember that, for any $\alpha \in \Omega^p(M)$, we have, setting $\alpha_H = \alpha \circ \Pi$ where $\Pi : TM \to H$ is the canonical projection and $\alpha_\xi = \theta \wedge i(\xi)\alpha$, the splitting $\alpha = \alpha_H + \alpha_\xi$ (cf. [15,16]). It follows the decomposition of $\Omega^*(M)$ as

$$\Omega^*(M) = \Omega^*_H(M) \oplus \theta \wedge \Omega^*_H(M),$$

where $\Omega^*_H(M)$ is the bundle of horizontal forms. Also, we recall the decomposition of any horizontal 2-tensor μ_H into $\mu_H = \mu_{H_+} + \mu_{H_-}$, where $\mu_{H_\pm} := \frac{1}{2}(\mu_H \pm \mu_H \circ J)$ are respectively the J-invariant part and the J-anti-invariant part of μ_H.

In the following, the torsion and the curvature of a connection ∇ are respectively defined by $T(X,Y) = [X,Y] - \nabla_X Y + \nabla_Y X$ and $R(X,Y) = [\nabla_Y, \nabla_X] - \nabla_{[Y,X]}$.

Proposition 2.1 (Generalized Tanaka–Webster connection, cf. [17,18,20]). *Let $(M, \theta, \xi, J, g_\theta)$ be contact metric manifold, then there exists a unique affine connection ∇ on TM with torsion T (called the generalized Tanaka–Webster connection) such that*:

(a) $\nabla\theta = 0$, $\nabla\xi = 0$,
(b) $\nabla g_\theta = 0$,
(c) $T_H = -d\theta \otimes \xi$ *and* $i(\xi)T = \frac{1}{2}(J \circ \mathcal{L}_\xi J)$,
(d) $g_\theta((\nabla_X J)(Y), Z) = \frac{1}{2} d\theta(X, N_H(Y,Z))$ *for any* $X, Y, Z \in TM$,

where $(\mathcal{L}_\xi J)(X) = [\xi, JX] - J[\xi, X]$ *and* $N(Y,Z) = J^2[Y,Z] + [JY, JZ] - J[Y, JZ] - J[JY, Z] + d\theta(Y,Z)\xi$.

The endomorphism $\tau := i(\xi)T$ is called the generalized Tanaka–Webster torsion. Note that τ is g_θ-symmetric with trace-free and satisfies $\tau(JX) = -J(\tau(X))$.

The curvature R of ∇ satisfies the following Bianchi identities (cf. [5,17]):

$$R_H(X,Y)Z + R_H(Z,X)Y + R_H(Y,Z)X = \omega_\theta(X,Y)\tau(Z) + \omega_\theta(Z,X)\tau(Y) + \omega_\theta(Y,Z)\tau(X), \quad (1)$$

$$R(X,\xi)Z + R(\xi,Z)X = (\nabla_X\tau)(Z) - (\nabla_Z\tau)(X),$$

$$\begin{aligned} g_\theta\big(R_H(X,Y)Z, W\big) - g_\theta\big(R_H(Z,W)X, Y\big) &= \omega_\theta(Y,Z) g_\theta\big(\tau(X), W\big) + \omega_\theta(X,W) g_\theta\big(\tau(Y), Z\big) \\ &\quad - \omega_\theta(X,Z) g_\theta\big(\tau(Y), W\big) - \omega_\theta(Y,W) g_\theta\big(\tau(X), Z\big), \end{aligned}$$

$$g_\theta\bigl(R(X,\xi)Z,W\bigr)=g_\theta\bigl((\nabla_W\tau)(X),Z\bigr)-g_\theta\bigl((\nabla_Z\tau)(X),W\bigr),$$

where $\omega_\theta := d\theta$, $(\nabla_X\tau)(Z)=\nabla_X\tau(Z)-\tau(\nabla_X Z)$ and $X,Y,Z,W\in H$.

The generalized Tanaka–Webster Ricci endomorphism Ric is given by

$$\mathrm{Ric}(X)=\mathrm{trace}_{g_\theta} R(.\,,X).$$

Notes that $\mathrm{Ric}(\xi)=\delta\tau$ with $\delta\tau=-\mathrm{trace}_{g_\theta}(\nabla_.\tau)(.)$.

The generalized Tanaka–Webster scalar curvature is $s=\mathrm{trace}_{g_\theta}(\mathrm{Ric})$.

A contact metric manifold $(M,\theta,\xi,J,g_\theta)$ for which J is integrable (i.e., $\nabla J=0$) is called a strictly pseudoconvex *CR* manifold.

The following identities hold for a strictly pseudoconvex *CR* manifold. For any $X,Y,Z\in H$,

$$\mathrm{Ric}_{H_-}(X,Y)=(d-1)\omega_\theta\bigl(\tau(X),Y\bigr), \tag{2}$$

$$\begin{aligned}&R_{H_-}(X,Y)Z+R_{H_-}(Z,X)Y+R_{H_-}(Y,Z)X\\&\quad=\omega_\theta(X,Y)\tau(Z)+\omega_\theta(Z,X)\tau(Y)+\omega_\theta(Y,Z)\tau(X).\end{aligned} \tag{3}$$

If $(M,\theta,\xi,J,g_\theta)$ is a strictly pseudoconvex *CR* manifold, the symmetric 2-tensor Ric_{H_+} is called the pseudo-Hermitian Ricci tensor, the 2-form $\rho_H(X,Y)=\mathrm{Ric}_{H_+}(X,JY)$ is called the pseudo-Hermitian Ricci form, and $\mathrm{trace}_{g_{\theta,H}}\mathrm{Ric}_{H_+}$ is called the pseudo-Hermitian scalar curvature. Note that the pseudo-Hermitian scalar curvature coincides with the generalized Tanaka–Webster scalar curvature.

3. Spin^c-structures and Dirac operators on contact metric manifolds

In this section we will describe the special features of the Spin (Spin^c)-structures on a contact metric manifold and their associated spinor bundles. For the definitions and more details about these notions we refer to [7].

3.1. Spin^c*-structures on contact metric manifolds*

Let $(M,\theta,\xi,J,g_\theta)$ be a contact metric manifold and $\mathcal{V}_\mathbb{C}$ the subbundle of $T^\mathbb{C}M$ given by $\mathcal{V}_\mathbb{C}=\mathbb{C}\xi$. Then, we have

$$T^\mathbb{C}M=T^{1,0}M\oplus T^{0,1}M\oplus\mathcal{V}_\mathbb{C},$$

where $T^{1,0}M$ (resp. $T^{0,1}M$) is the subbundle given by the eigenspace of the complex extension of J on $H^\mathbb{C}$ to the eigenvalue $\sqrt{-1}$ (resp. $-\sqrt{-1}$). We have an isomorphism

$$\wedge^m T^\mathbb{C}M=\bigoplus_{p+q+r=m}\wedge^p T^{1,0}M\otimes\wedge^q T^{0,1}M\otimes\wedge^r\langle\xi\rangle,$$

where we observe that $\wedge^r\langle\xi\rangle=0$ if $r>1$. We set

$$\wedge_H^{p,q}(M):=\wedge^p T^{1,0}M^*\otimes\wedge^q T^{0,1}M^*,\qquad \wedge^{p,0}(M):=\wedge_H^{p,0}(M)\oplus\wedge_H^{p-1,0}(M)\wedge\theta.$$

The bundle $K:=\wedge^{d+1,0}(M)\simeq\wedge_H^{d,0}(M)$ and its dual, denoted K^{-1}, are respectively called the canonical and the anticanonical bundle. In the following, we set $\Omega_H^{p,q}(M)=\Gamma(\wedge_H^{p,q}(M))$.

A contact metric structure on M^{2d+1} produces a reduction of the $SO(2d+1)$-structure of TM to the subgroup $U(d)$-structure, where $U(d)$ is viewed canonically as a subgroup of $SO(2d+1)$. Arguing exactly as in Corollary 3.4.5 of [9], we deduce the following result:

Proposition 3.1. *Any contact metric manifold $(M, \theta, \xi, J, g_\theta)$ admits a* Spin^c*-structure whose determinant line bundle is K^{-1}.*

Proposition 3.2. *The spinor bundle ΣM^c can be identified with the bundle $\wedge_H^{0,*}(M)$ of the $(0,*)$ forms and the Clifford multiplication by a vector field X on a $(0,q)$ form α is given by*

$$X.\alpha = \sqrt{2}\big((X_H^{0,1})^* \wedge \alpha - i(X_H^{0,1})\alpha\big) + (-1)^{q+1}\sqrt{-1}\,\theta(X)\alpha,$$

where X_H denotes the horizontal part of X and i is the contraction operator.

The proof of this proposition is based on the following lemma (as in Corollary 3.4.6 of [9]).

Lemma 3.1. *For the above defined multiplication, we have*

$$X.X.\alpha = -g_\theta(X,X)\alpha$$

and

$$(\sqrt{-1})^{d+1} v_{g_\theta}.\alpha = \alpha,$$

where v_{g_θ} is the canonical volume element (*i.e.*, $v_{g_\theta} = \frac{1}{d!}\theta \wedge (d\theta)^d$).

Proof. By the above formula, we have

$$\begin{aligned}
X.X.\alpha &= \sqrt{2}X.\big((X_H^{0,1})^* \wedge \alpha\big) - \sqrt{2}X.\big(i(X_H^{0,1})\alpha\big) + (-1)^{q+1}\sqrt{-1}\,\theta(X)X.\alpha \\
&= -2i(X_H^{0,1})\big((X_H^{0,1})^* \wedge \alpha\big) + (-1)^{q+2}\sqrt{-1}\sqrt{2}\,\theta(X)\big((X_H^{0,1})^* \wedge \alpha\big) \\
&\quad - 2(X_H^{0,1})^* \wedge i(X_H^{0,1})\alpha + (-1)^{q+1}\sqrt{-1}\sqrt{2}\,\theta(X)i(X_H^{0,1})\alpha \\
&\quad + (-1)^{q+1}\sqrt{-1}\sqrt{2}\,\theta(X)\big((X_H^{0,1})^* \wedge \alpha\big) - (-1)^{q+1}\sqrt{-1}\sqrt{2}\,\theta(X)i(X_H^{0,1})\alpha - \theta^2(X)\alpha \\
&= -2|X_H^{0,1}|^2\alpha - \theta^2(X)\alpha.
\end{aligned}$$

Since $|X_H^{0,1}|^2 = \frac{1}{2}|X_H|^2$, we have $X.X.\alpha = -g_\theta(X,X)\alpha$. Now, for any $X \in H$, we have $JX^{0,1} = -\sqrt{-1}X^{0,1}$. Since the duality is antilinear, we deduce that

$$JX.\alpha = \sqrt{2}\sqrt{-1}\big((X^{0,1})^* \wedge \alpha + i(X^{0,1})\alpha\big).$$

Hence, we have

$$\begin{aligned}
X.JX.\alpha &= \sqrt{2}\sqrt{-1}X.\big((X^{0,1})^* \wedge \alpha + i(X^{0,1})\alpha\big) \\
&= \sqrt{-1}\big(-2|X^{0,1}|^2\alpha + 4(X^{0,1})^* \wedge i(X^{0,1})\alpha\big) \\
&= -\sqrt{-1}\big(|X|^2\alpha - 4(X^{0,1})^* \wedge i(X^{0,1})\alpha\big).
\end{aligned}$$

Now, let $\{\varepsilon_1, J\varepsilon_1, \dots, \varepsilon_d, J\varepsilon_d, \xi\}$ a local orthonormal frame, where $\{\varepsilon_1, J\varepsilon_1, \dots, \varepsilon_d, J\varepsilon_d\}$ is a local orthonormal frame of H, then we have

$$\varepsilon_i.J\varepsilon_i.\alpha = -\sqrt{-1}\big(\alpha - 4\overline{Z}_i^* \wedge i(\overline{Z}_i)\alpha\big),$$

with $\overline{Z}_i = \frac{1}{2}(\varepsilon_i + \sqrt{-1}J\varepsilon_i)$. Let $\alpha = \overline{Z}_{j_1}^* \wedge \overline{Z}_{j_2}^* \wedge \dots \wedge \overline{Z}_{j_q}^*$. If $i \neq \{j_1, j_2, \dots, j_q\}$, then (cf. Corollary 3.4.5 of [9])

$$\overline{Z}_i^* \wedge i(\overline{Z}_i)\alpha = 0 \quad \text{and} \quad \varepsilon_i.J\varepsilon_i.\alpha = -\sqrt{-1}\,\alpha,$$

whereas if $i \in \{j_1, j_2, \dots, j_q\}$, then

$$\overline{Z}_i^* \wedge i(\overline{Z}_i)\alpha = \frac{1}{2}\alpha \quad \text{and} \quad \varepsilon_i.J\varepsilon_i.\alpha = \sqrt{-1}\,\alpha.$$

Now, since $v_{g_\theta} = \xi.\varepsilon_1.J\varepsilon_1 \dots \varepsilon_d.J\varepsilon_d$, then

$$v_{g_\theta}.\alpha = (-\sqrt{-1})^{d-q}(\sqrt{-1})^{q}(-1)^{q+1}\sqrt{-1}\,\alpha = (\sqrt{-1})^{d+1}(-1)^{d+1}\alpha.$$

Hence, $(\sqrt{-1})^{d+1} v_{g_\theta}.\alpha = \alpha$. □

Proof of Proposition 3.2. The above action satisfies the Clifford relation and consequently extends by complex linear endomorphisms to an action of $\mathbb{C}l(M)$. Since, at each point, $\dim(\wedge_H^{0,*}(M)) = 2^d$, hence $\wedge_H^{0,*}(M)$ is an irreductible module for $\mathbb{C}l(M)$. Now, since $(\sqrt{-1})^{d+1} v_{g_\theta} = \mathrm{Id}$ on $\wedge_H^{0,*}(M)$, we deduce that $\Sigma M^c = \wedge_H^{0,*}(M) = \bigoplus_q \wedge_H^{0,q}(M)$. □

In the following we call the above Spin^c-structure the canonical Spin^c-structure of the contact metric manifold $(M, \theta, \xi, J, g_\theta)$. Now, consider any other Spin^c-structure on (M, g_θ) with determinant line bundle L. Then, the associated principal $Spin^{\mathbb{C}}$-bundle differs from the canonical principal $Spin^{\mathbb{C}}$-bundle by tensoring with some $U(1)$-principal bundle. If $\mathcal{L}$ is the complex line bundle associated to this $U(1)$-principal bundle, then the spinor bundle is $\Sigma M^c = \wedge_H^{0,*}(M) \otimes \mathcal{L}$. Note that $\mathcal{L}^2 = K \otimes L$.

For any spinor bundle ΣM^c associated to a Spin^c-structure on (M, g_θ), the Clifford multiplications by $\sqrt{-1}\xi$ and $\sqrt{-1}\omega_\theta$ respectively induce the decompositions (cf. [1,10,13]):

$$\Sigma M^c = \Sigma^+ M^c \oplus \Sigma^- M^c,$$

and

$$\Sigma M^c = \bigoplus_{q=0}^{d} \Sigma_{(d-2q)} M^c,$$

where $\Sigma^\pm M^c$ is the eigenspace of $\sqrt{-1}\,\xi$ to the eigenvalue ± 1 and $\Sigma_k M^c$ is the eigenspace of $\sqrt{-1}\,\omega_\theta$ to the eigenvalue k. By Proposition 3.2, we have $\Sigma^\pm M^c \simeq \wedge_H^{0,\mathrm{even/odd}}(M) \otimes \mathcal{L}$ and $\Sigma_{(d-2q)} M^c \simeq \wedge_H^{0,q}(M) \otimes \mathcal{L}$.

3.2. *Spinorial connections and Dirac type operators on contact metric manifolds*

Let (M, g_θ) be a contact metric manifold endowed with a Spin^c-structure. Each unitary connection A on L together with the generalized Tanaka–Webster connection ∇ induce a spinorial connection ∇^A

on ΣM^c. The Dirac operator associated will be denoted by $\mathcal{D}_A$. The Kohn–Dirac operator $\mathcal{D}_H^A$ is the differential operator defined by:

$$\mathcal{D}_H^A = \sum_i \varepsilon_i . \nabla^A_{\varepsilon_i},$$

where $\{\varepsilon_i\}$ is a local orthonormal frame of H.

Proposition 3.3. *The operators $\mathcal{D}_A$ and $\mathcal{D}_H^A$ satisfy the identities*:

$$-\frac{1}{2}\{\mathcal{D}_A, \lambda_\xi\} = \nabla^A_\xi, \tag{4}$$

$$\frac{1}{2}\lambda_\xi \circ [\mathcal{D}_A, \lambda_\xi] = \mathcal{D}_H^A, \tag{5}$$

$$\frac{1}{2}\lambda_\xi \circ [\mathcal{D}_A^2, \lambda_\xi] = \{\mathcal{D}_H^A, \lambda_\xi \circ \nabla^A_\xi\}, \tag{6}$$

$$-\frac{1}{2}\lambda_\xi \circ \{\mathcal{D}_A^2, \lambda_\xi\} = \mathcal{D}_H^{A^2} - \nabla^{A^2}_{\xi,\xi}, \tag{7}$$

where $\lambda_.$ is the endomorphism of $\Gamma(\Sigma M^c)$ given by the Clifford product by a form or a vector field and $[\ ,\ ]$ (*resp.* $\{\ ,\ \}$) *is the commutator* (*resp. anticommutator*).

Proof. We have

$$\mathcal{D}_A = \mathcal{D}_H^A + \lambda_\xi \circ \nabla^A_\xi,$$

with $\mathcal{D}_H^A$ (resp. $\lambda_\xi \circ \nabla^A_\xi$) anticommutes (resp. commutes) with λ_ξ (since λ_ξ is ∇^A-parallel). Hence,

$$\{\mathcal{D}_A, \lambda_\xi\} = \{\lambda_\xi \circ \nabla^A_\xi, \lambda_\xi\} = -2\nabla^A_\xi,$$

and

$$[\mathcal{D}_A, \lambda_\xi] = [\mathcal{D}_H^A, \lambda_\xi] = -2\lambda_\xi \circ \mathcal{D}_H^A.$$

Now, we have

$$\mathcal{D}_A^2 = (\mathcal{D}_H^A + \lambda_\xi \circ \nabla^A_\xi) \circ (\mathcal{D}_H^A + \lambda_\xi \circ \nabla^A_\xi) = \mathcal{D}_H^{A^2} - \nabla^{A^2}_{\xi,\xi} + \{\mathcal{D}_H^A, \lambda_\xi \circ \nabla^A_\xi\},$$

with $\mathcal{D}_H^{A^2} - \nabla^{A^2}_{\xi,\xi}$ (resp. $\{\mathcal{D}_H^A, \lambda_\xi \circ \nabla^A_\xi\}$) commutes (resp. anticommutes) with λ_ξ. The last two equations follow directly. $\square$

Let $\overline{\nabla}^A$ and $\overline{\mathcal{D}}_A$ be the spinorial connection and the Dirac operator on ΣM^c induced by the Levi-Civita connection on TM together with a unitary connection A on L. We have

Proposition 3.4. *For any $\psi \in \Gamma(\Sigma M^c)$, we have*

$$\nabla^A_X \psi - \overline{\nabla}^A_X \psi = \frac{1}{4} JX.\xi.\psi - \frac{1}{4}\theta(X)\omega_\theta.\psi - \frac{1}{2}\tau(X).\xi.\psi,$$

$$\mathcal{D}_A \psi - \overline{\mathcal{D}}_A \psi = \frac{1}{4}\xi.\omega_\theta.\psi.$$

Moreover, if M is spin, we have

$$\nabla_\xi \psi = \mathcal{L}_\xi \psi,$$

where $\mathcal{L}_\xi$ is the metric Lie derivative associated to the Webster metric (*cf.* [3]).

Proof. The calculation is local and locally the Spin^c-spinor bundle can be decomposed as $\Sigma M^c = \Sigma M \otimes L^{1/2}$, where ΣM is the local Spin-spinor bundle and $L^{1/2}$ is a local square root of L. Also it is sufficient to verify the formula for a local section of ΣM. The difference between the generalized Tanaka–Webster connection ∇ and the Levi-Civita connection $\overline{\nabla}$ is given by (cf. [4]):

$$\nabla - \overline{\nabla} = -\frac{1}{2}\theta \odot J + \left(\frac{1}{2}\omega_\theta - A_\theta\right) \otimes \xi + \tau \otimes \theta,$$

where $\odot$ denotes the symmetric product (i.e., for any $X, Y \in TM$, $(\theta \odot J)(X, Y) = \theta(X)JY + \theta(Y)JX$). Now, let $\{e_i\}$ a local g_θ-orthonormal frame and $\{\psi_\alpha\}$ a local spinorial frame, then

$$\nabla_X \psi_\alpha - \overline{\nabla}_X \psi_\alpha = -\frac{1}{2} \sum_{1 \leqslant i < j \leqslant 2d+1} g_\theta(\nabla_X e_j - \overline{\nabla}_X e_j, e_i) e_i . e_j . \psi_\alpha.$$

In a local orthonormal frame $\{\varepsilon_1, \dots, \varepsilon_{2d}, \xi\}$, where $\{\varepsilon_i\}$, $i \leqslant 2d$, is a local orthonormal frame of H, we have

$$\begin{aligned}
\nabla_X \psi_\alpha - \overline{\nabla}_X \psi_\alpha &= -\frac{1}{2} \sum_{1 \leqslant i < j \leqslant 2d} g_\theta\big((\nabla_X - \overline{\nabla}_X)\varepsilon_j, \varepsilon_i\big)\varepsilon_i . \varepsilon_j . \psi_\alpha + \frac{1}{2} \sum_{1 \leqslant i \leqslant 2d} g_\theta(\overline{\nabla}_X \xi, \varepsilon_i)\varepsilon_i . \xi . \psi_\alpha \\
&= \frac{1}{4} \sum_{1 \leqslant i < j \leqslant 2d} \omega_\theta(\varepsilon_j, \varepsilon_i)\varepsilon_i . \varepsilon_j . \psi_\alpha + \frac{1}{4} \sum_{1 \leqslant i \leqslant 2d} \omega_\theta(X, \varepsilon_i)\varepsilon_i . \xi . \psi_\alpha \\
&\quad - \frac{1}{2} \sum_{1 \leqslant i \leqslant 2d} g_\theta\big(\tau(X), \varepsilon_i\big)\varepsilon_i . \xi . \psi_\alpha \\
&= \frac{1}{4} JX . \xi . \psi_\alpha - \frac{1}{4}\theta(X)\omega_\theta . \psi_\alpha - \frac{1}{2}\tau(X) . \xi . \psi_\alpha.
\end{aligned}$$

By taking the trace, we obtain

$$\begin{aligned}
\mathcal{D}\psi - \overline{\mathcal{D}}\psi &= \frac{1}{4} \sum_{i \leqslant 2d+1} e_i . J e_i . \xi . \psi - \frac{1}{4} \sum_{i \leqslant 2d+1} \theta(e_i) e_i . \omega_\theta . \psi - \frac{1}{2} \sum_{i \leqslant 2d+1} e_i . \tau(e_i) . \xi . \psi \\
&= \frac{1}{2}\omega_\theta . \xi . \psi - \frac{1}{4}\xi . \omega_\theta . \psi = \frac{1}{4}\xi . \omega_\theta . \psi.
\end{aligned}$$

Now, we deduce from the formula $\nabla_\xi \psi = \overline{\nabla}_\xi \psi - \frac{1}{4}\omega_\theta . \psi$, together with Proposition 17 of [3] that $\nabla_\xi \psi$ coincides with $\mathcal{L}_\xi \psi$. $\square$

Remark 3.1. As a consequence of the previous proposition, we obtain that if M is compact, then the Dirac operator $\mathcal{D}_A$ is formally self-adjoint for the natural inner product on $\Gamma(\Sigma M^c)$. A such Dirac operator is called a nice geometric Dirac operator by Nicolaescu [13].

Now, suppose that M is a strictly pseudoconvex CR manifold endowed with a Spin^c-structure. Let $\Sigma M^c = \wedge_H^{0,*}(M) \otimes \mathcal{L}$ be the spinor bundle (with $\mathcal{L}$ as above). To each unitary connection A on L, correspond a unitary connection $\mathcal{A}$ on $\mathcal{L}$. The correspondence is given by $\mathcal{A}^2 = A_c \otimes A$, where A_c is the Webster connection on K (cf. [1]). The associated covariant derivative on $\mathcal{L}$ is denoted by $\nabla^{\mathcal{A}}$. Let $\Omega_H^{0,*}(M;\mathcal{L}) = \Gamma(\wedge_H^{0,*}(M) \otimes \mathcal{L})$, we define, $\nabla_W^{\mathcal{A}^q} : \Omega_H^{0,q}(M;\mathcal{L}) \to \Omega_H^{0,q}(M;\mathcal{L})$ by the usual rule

$$\nabla_W^{\mathcal{A}^q}(\alpha \otimes z) = (\nabla_W^q \alpha) \otimes z + \alpha \otimes \nabla_W^{\mathcal{A}} z,$$

where $\nabla_W^q \alpha$ is the natural extension of the generalized Tanaka–Webster connection to $\Omega_H^{0,q}(M)$, $z \in \Gamma(\mathcal{L})$ and $W \in T^{\mathbb{C}}M$.

Let $\bar\partial_H^{\mathcal{A}^q} : \Omega_H^{0,q}(M;\mathcal{L}) \to \Omega_H^{0,q+1}(M;\mathcal{L})$ (resp. $\bar\partial_H^{\mathcal{A}^{q*}} : \Omega_H^{0,q}(M;\mathcal{L}) \to \Omega_H^{0,q-1}(M;\mathcal{L})$) given by

$$\bar\partial_H^{\mathcal{A}^q} = \sum_{i=1}^{d} \overline{Z}_i^* \wedge \nabla_{\overline{Z}_i}^{\mathcal{A}^q} \quad \text{(resp. } \bar\partial_H^{\mathcal{A}^{q*}} = -\sum_{i=1}^{d} i(\overline{Z}_i)\nabla_{Z_i}^{\mathcal{A}^q}\text{)}.$$

It follows from Proposition 3.2, that we have on $\Omega_H^{0,*}(M;\mathcal{L})$

$$\nabla_W^A = \sum_{q=0}^{d} \nabla_W^{\mathcal{A}^q}, \qquad \mathcal{D}_H^A = \sqrt{2}\sum_{q=0}^{d}(\bar\partial_H^{\mathcal{A}^q} + \bar\partial_H^{\mathcal{A}^{q*}}),$$

and

$$\lambda_\xi \circ \nabla_\xi^A = \sum_{q=0}^{d} (-1)^{q+1}\sqrt{-1}\nabla_\xi^{\mathcal{A}^q}.$$

Remark 3.2. For the canonical Spin^c-structure, the operator $\bar\partial_H^{\mathcal{A}}$ coincides with the usual operator $\bar\partial_H$ and $\mathcal{D}_H^{A_{\mathrm{ac}}^2} = \sum_{q=0}^{d} \square_H^q$, where $\square_H^q$ is the Kohn Laplacian on $(0,q)$-forms (i.e., $\square_H^q = 2(\bar\partial_H^{(q+1)^*}\bar\partial_H^q + \bar\partial_H^{(q-1)}\bar\partial_H^{q^*})$).

4. Lichnerowicz type formulas and vanishing theorems on Spin^c contact metric manifolds

4.1. Lichnerowicz type formulas

Let (M, g_θ, ∇) be an $(2d+1)$-dimensional contact metric manifold endowed with the Webster metric and with the generalized Tanaka–Webster connection and E be a Riemannian vector bundle over M. Denote by $\Omega^*(M;E)$ the bundle of E-valued forms on M. Remember (cf. [14,19]) that for any $\alpha \in \Omega^p(M;E)$, the covariant derivative and the divergence of α are respectively given by:

$$(\nabla_X\alpha)(X_1,\dots,X_p) = \nabla_X^E\alpha(X_1,\dots,X_p) - \sum_{i=1}^{p}\alpha(X_1,\dots,\nabla_X X_i,\dots,X_p),$$

$$(\delta\alpha)(X_1,\dots,X_{p-1}) = -\operatorname{trace}_{g_\theta}(\nabla_{.}\,\alpha)(\,.\,,X_1,\dots,X_{p-1}).$$

For any $\alpha_H \in \Omega^p_H(M; E)$, we define

$$(\delta_H \alpha_H)(X_1, \dots, X_{p-1}) = -\operatorname{trace}_{g_{\theta,H}}(\nabla_. \alpha_H)(. , X_1, \dots, X_{p-1}),$$

and for $p \geqslant 2$

$$(\wedge_H \alpha_H)(X_1, \dots, X_{p-2}) = \frac{1}{2}\operatorname{trace}_{g_{\theta,H}} \omega_\theta(. , .)\alpha_H(. , . , X_1, \dots, X_{p-2}),$$

with $X_i \in H$.

Let S be a Dirac bundle over M. For any $Q \in \Omega^2(M; \wedge^2(M))$ and $\alpha \in \Omega^1(M; \operatorname{End}(S))$, we define $Q(\alpha) \in \Omega^1(M; \operatorname{End}(S))$ by

$$Q(\alpha)(X) = \sum_i Q(e_i, X).\alpha(e_i) = \frac{1}{2}\sum_{i,j,k}\big(Q(e_i, X)\big)(e_j, e_k)e_j.e_k.\alpha(e_i),$$

where $\{e_i\}$ is a local g_θ-orthonormal frame of TM.

In the following, the connections (resp. curvatures) on $\mathrm{Cl}(M)$ or S will be denoted by ∇ (resp. R).

Proposition 4.1 (Mok–Siu–Yeung type formulas).

(i) *If $\delta_H Q_H = 0$, then we have*

$$\delta_H\big(Q_H(\nabla)\big) = \mathcal{R}^Q_H - \lambda_{(\wedge_H Q_H)} \circ \nabla_\xi, \tag{8}$$

where $\mathcal{R}^Q_H$ is the endomorphism given in a local orthonormal frame $\{\varepsilon_i\}$ of H by

$$\mathcal{R}^Q_H = -\frac{1}{2}\sum_{i,j} Q_H(\varepsilon_i, \varepsilon_j).R_H(\varepsilon_i, \varepsilon_j).$$

(ii) *If $\delta Q_\xi = 0$, then we have*

$$\delta\big(Q_\xi(\nabla)\big) = \mathcal{R}^Q_\xi + \mathcal{D}^Q_\tau, \tag{9}$$

where $\mathcal{R}^Q_\xi$ (resp. $\mathcal{D}^Q_\tau$) is the endomorphism (resp. differential operator) given by

$$\mathcal{R}^Q_\xi = -\sum_i Q(\xi, \varepsilon_i).R(\xi, \varepsilon_i) \quad \Big(\textit{resp. } \mathcal{D}^Q_\tau = \sum_i Q(\xi, \varepsilon_i).\nabla_{\tau(\varepsilon_i)}\Big).$$

Proof. Let $\{\varepsilon_1, \dots, \varepsilon_{2d}, \xi\}$ be a local orthonormal frame, where $\{\varepsilon_i\}$, $i \leqslant 2d$, is a local orthonormal frame of H. For any $X \in H$, we have

$$\big(Q_H(\nabla)\big)(X) = \sum_i Q_H(\varepsilon_i, X).\nabla_{\varepsilon_i}.$$

We obtain

$$\begin{aligned}\delta_H\big(Q_H(\nabla)\big) &= -\sum_j\big(\nabla_{\varepsilon_j} Q_H(\nabla)\big)(\varepsilon_j) = -\sum_j\Big(\nabla_{\varepsilon_j}\big(Q_H(\nabla)\big)(\varepsilon_j) - \big(Q_H(\nabla)\big)(\nabla_{\varepsilon_j}\varepsilon_j)\Big)\\ &= -\sum_{i,j}\Big(\nabla_{\varepsilon_j}\big(Q_H(\varepsilon_i, \varepsilon_j).\nabla_{\varepsilon_i}\big) - Q_H(\varepsilon_i, \nabla_{\varepsilon_j}\varepsilon_j).\nabla_{\varepsilon_i}\Big)\end{aligned}$$

$$
\begin{aligned}
&= -\sum_{i,j}\big(\big(\nabla_{\varepsilon_j} Q_H(\varepsilon_i,\varepsilon_j)\big).\nabla_{\varepsilon_i} + Q_H(\varepsilon_i,\varepsilon_j).\nabla_{\varepsilon_j}\nabla_{\varepsilon_i} - Q_H(\varepsilon_i,\nabla_{\varepsilon_j}\varepsilon_j).\nabla_{\varepsilon_i}\big)\\
&= -\sum_{i,j}\big((\nabla_{\varepsilon_j} Q_H)(\varepsilon_i,\varepsilon_j).\nabla_{\varepsilon_i} + Q_H(\varepsilon_i,\varepsilon_j).\nabla_{\varepsilon_j}\nabla_{\varepsilon_i} + Q_H(\nabla_{\varepsilon_j}\varepsilon_i,\varepsilon_j).\nabla_{\varepsilon_i}\big)\\
&= -\sum_{i,j}(\nabla_{\varepsilon_j} Q_H)(\varepsilon_i,\varepsilon_j).\nabla_{\varepsilon_i} - \sum_{i,j} Q_H(\varepsilon_i,\varepsilon_j).(\nabla_{\varepsilon_j}\nabla_{\varepsilon_i} - \nabla_{\nabla_{\varepsilon_j}\varepsilon_i})\\
&= -\sum_{i}(\delta_H Q_H)(\varepsilon_i).\nabla_{\varepsilon_i} - \sum_{i,j} Q_H(\varepsilon_i,\varepsilon_j).\nabla^2_{\varepsilon_j,\varepsilon_i}\\
&= -\sum_{i}(\delta_H Q_H)(\varepsilon_i).\nabla_{\varepsilon_i} - \frac{1}{2}\sum_{i,j} Q_H(\varepsilon_i,\varepsilon_j).(\nabla^2_{\varepsilon_j,\varepsilon_i} - \nabla^2_{\varepsilon_i,\varepsilon_j}).
\end{aligned}
$$

Now, for any $X, Y \in H$, we have

$$\nabla^2_{Y,X} - \nabla^2_{X,Y} = R_H(X,Y) - \nabla_{T_H(X,Y)} = R_H(X,Y) + \omega_\theta(X,Y)\nabla_\xi.$$

The assumption $\delta_H Q_H = 0$ together with the previous formula yield

$$
\begin{aligned}
\delta_H\big(Q_H(\nabla)\big) &= -\frac{1}{2}\sum_{i,j} Q_H(\varepsilon_i,\varepsilon_j).R_H(\varepsilon_i,\varepsilon_j) - \frac{1}{2}\sum_{i,j}\omega_\theta(\varepsilon_i,\varepsilon_j)Q_H(\varepsilon_i,\varepsilon_j).\nabla_\xi\\
&= \mathcal{R}^Q_H - (\wedge_H Q_H).\nabla_\xi.
\end{aligned}
$$

For any $X \in TM$, we have

$$\big(Q_\xi(\nabla)\big)(X) = \sum_i Q_\xi(\varepsilon_i, X).\nabla_{\varepsilon_i} + Q_\xi(\xi, X).\nabla_\xi = -\theta(X)\sum_i Q(\xi,\varepsilon_i).\nabla_{\varepsilon_i} + Q(\xi, X).\nabla_\xi.$$

We obtain

$$
\begin{aligned}
\delta\big(Q_\xi(\nabla)\big) &= -\sum_j\big(\nabla_{\varepsilon_j} Q_\xi(\nabla)\big)(\varepsilon_j) - \big(\nabla_\xi Q_\xi(\nabla)\big)(\xi)\\
&= -\sum_j\big(\nabla_{\varepsilon_j}\big(Q_\xi(\nabla)\big)(\varepsilon_j) - \big(Q_\xi(\nabla)\big)(\nabla_{\varepsilon_j}\varepsilon_j)\big) - \nabla_\xi\big(Q_\xi(\nabla)\big)(\xi)\\
&= -\sum_j\big(\nabla_{\varepsilon_j}\big(Q(\xi,\varepsilon_j).\nabla_\xi\big) - Q(\xi,\nabla_{\varepsilon_j}\varepsilon_j).\nabla_\xi\big) + \sum_j \nabla_\xi\big(Q(\xi,\varepsilon_j).\nabla_{\varepsilon_j}\big)\\
&= -\sum_j\big(\big(\nabla_{\varepsilon_j} Q(\xi,\varepsilon_j)\big).\nabla_\xi + Q(\xi,\varepsilon_j).\nabla_{\varepsilon_j}\nabla_\xi - Q(\xi,\nabla_{\varepsilon_j}\varepsilon_j).\nabla_\xi\big)\\
&\quad + \sum_j\big(\big(\nabla_\xi Q(\xi,\varepsilon_j)\big).\nabla_{\varepsilon_j} + Q(\xi,\varepsilon_j).\nabla_\xi\nabla_{\varepsilon_j}\big)\\
&= -\sum_j\big(\nabla_{\varepsilon_j} i(\xi)Q\big)(\varepsilon_j).\nabla_\xi + \sum_j\big(\nabla_\xi i(\xi)Q\big)(\varepsilon_j).\nabla_{\varepsilon_j}\\
&\quad + \sum_j Q(\xi,\varepsilon_j).(\nabla_\xi\nabla_{\varepsilon_j} - \nabla_{\nabla_\xi\varepsilon_j}) - \sum_j Q(\xi,\varepsilon_j).\nabla_{\varepsilon_j}\nabla_\xi\\
&= -(\delta Q_\xi)(\xi).\nabla_\xi - \sum_j(\delta Q_\xi)(\varepsilon_j).\nabla_{\varepsilon_j} - \sum_j Q(\xi,\varepsilon_j).(\nabla^2_{\varepsilon_j,\xi} - \nabla^2_{\xi,\varepsilon_j}).
\end{aligned}
$$

Using the relation $\nabla^2_{X,\xi} - \nabla^2_{\xi,X} = R(\xi, X) - \nabla_{\tau(X)}$, $(X \in H)$, and $\delta Q_\xi = 0$, we obtain

$$\delta\big(Q_\xi(\nabla)\big) = -\sum_i Q(\xi, \varepsilon_i).R(\xi, \varepsilon_i) + \sum_i Q(\xi, \varepsilon_i).\nabla_{\tau(\varepsilon_i)} = \mathcal{R}^Q_\xi + \mathcal{D}^Q_\tau. \qquad \square$$

Remark 4.1. The classical example of $\wedge^2(M)$-valued 2-form on a Riemannian manifold M is the curvature tensor. Moreover, on a locally symmetric space the curvature tensor is parallel for the Levi-Civita connection. In [11] and [21], Mok, Siu and Yeung derive similar formulas to (8) for harmonic maps and forms defined on symmetric spaces. As applications they have obtained rigidity theorems for the symmetric spaces. Recently, by a Riemannian analogue of (8), Friedrich and Kirchberg [6] have obtained estimates on the first eigenvalue of the Dirac operator on Riemannian manifolds with harmonic curvature tensor.

A horizontal 2-form α_H such that $\wedge_H \alpha_H = 0$ is called a primitive 2-form. For any horizontal 2-form α_H, we associate a primitive 2-form denoted α_{H_0} by setting

$$\alpha_{H_0} = \alpha_H - \frac{1}{d}\omega_\theta \otimes (\wedge_H \alpha_H).$$

Now, let $\mathcal{D}^A_{H,J}$, $\mathcal{D}^A_{H,\tau}$ and $(\nabla^A_H)^*\nabla^A_H$ be the differential operators respectively given by

$$\mathcal{D}^A_{H,J} = \sum_i \varepsilon_i.\nabla^A_{J\varepsilon_i}, \qquad \mathcal{D}^A_{H,\tau} = \sum_i \varepsilon_i.\nabla^A_{\tau(\varepsilon_i)}, \qquad (\nabla^A_H)^*\nabla^A_H = -\sum_i \nabla^{A^2}_{\varepsilon_i,\varepsilon_i},$$

where $\{\varepsilon_i\}$ is a local orthonormal frame of H, and, $(\nabla^A_{1,0})^*\nabla^A_{1,0}$ (resp. $(\nabla^A_{0,1})^*\nabla^A_{0,1}$) the differential operator given by

$$(\nabla^A_{1,0})^*\nabla^A_{1,0} = -\sum_i \nabla^{A^2}_{\overline{Z}_i,Z_i} \quad \Big(\text{resp. } (\nabla^A_{0,1})^*\nabla^A_{0,1} = -\sum_i \nabla^{A^2}_{Z_i,\overline{Z}_i}\Big),$$

with $Z_i = \frac{1}{2}(e_i - \sqrt{-1}Je_i)$ where $\{e_1, Je_1, \ldots, e_d, Je_d\}$ is a local orthonormal frame of H.

Proposition 4.2 (Lichnerowicz type formulas). *Let Ω^A be the curvature 2-form on L associated to a connection A. Then the following identities hold*:

$$\{\mathcal{D}^A_H, \lambda_\xi \circ \nabla^A_\xi\} = \lambda_\xi \circ \mathcal{D}^A_{H,\tau} - \frac{1}{2}\lambda_{\xi.\delta\tau} + \frac{\sqrt{-1}}{2}\lambda_{\theta\wedge i(\xi)\Omega^A}, \tag{10}$$

$$\mathcal{D}^{A^2}_H = (\nabla^A_H)^*\nabla^A_H - \lambda_{\omega_\theta} \circ \nabla^A_\xi + \frac{1}{4}s + \frac{\sqrt{-1}}{2}\lambda_{\Omega^A_H}. \tag{11}$$

Moreover, in the strictly pseudoconvex case, we have

$$\begin{aligned}\mathcal{D}^{A^2}_H = {} & 2\left(1 + \frac{\sqrt{-1}}{d}\lambda_{\omega_\theta}\right)(\nabla^A_{0,1})^*\nabla^A_{0,1} + 2\left(1 - \frac{\sqrt{-1}}{d}\lambda_{\omega_\theta}\right)(\nabla^A_{1,0})^*\nabla^A_{1,0} \\ & + \frac{1}{4}\left(s - \frac{2}{d}\lambda_{\omega_\theta.\rho_H}\right) + \frac{\sqrt{-1}}{2}\lambda_{\Omega^A_{H_0}},\end{aligned} \tag{12}$$

$$\mathcal{D}^{A^2}_{H,J} - \mathcal{D}^{A^2}_H = -\sqrt{-1}\,\lambda_{\Omega^A_{H_-}}. \tag{13}$$

Remark 4.2. Similar identities to (10) and (11) have been obtained in dimension 3 by Mrowka, Ozsvath, Yu [10] and Nicolaescu [12,13]. Note that identity (12) is a spinorial analogue of the Weitzenbock–Tanaka formula on $\Omega_H^{0,q}(M)$ in [17].

Proof of Proposition 4.2. All these Lichnerowicz type formulas on ΣM^c come from formulas (8) and (9) for some particular curvature tensors on M. Consider the parallel $\wedge^2(M)$-valued 2-form Q^{can} defined by

$$Q^{\mathrm{can}}(X,Y)=X^*\wedge Y^*.$$

By taking $Q=Q^{\mathrm{can}}$ in (8) and (9), we obtain the formulas:

$$\mathcal{D}_H^{A^2}-(\nabla_H^A)^*\nabla_H^A=-\lambda_{\omega_\theta}\circ\nabla_\xi^A+\mathcal{R}_H^A$$

and

$$\{\mathcal{D}_H^A,\lambda_\xi\circ\nabla_\xi^A\}=\lambda_\xi\circ\mathcal{D}_{H,\tau}^A+\mathcal{R}_\xi^A,$$

where $\mathcal{R}_H^A$ and $\mathcal{R}_\xi^A$ are the endomorphisms locally given by:

$$\mathcal{R}_H^A=-\frac{1}{2}\sum_{i,j}\varepsilon_i.\varepsilon_j.R_H^A(\varepsilon_i,\varepsilon_j),\qquad \mathcal{R}_\xi^A=-\sum_i\xi.\varepsilon_i.R^A(\xi,\varepsilon_i).$$

Now, we have to calculate the curvature terms. Since the calculation is local, we may assume $\Sigma M^c=\Sigma M\otimes L^{1/2}$, where ΣM is the local spinor bundle and $L^{1/2}$ is a local square root of L. We furnish ΣM^c with the tensor product metric and tensor product connection. Hence, we have for any $\psi\in\Gamma(\Sigma M)$ and any $\sigma\in\Gamma(L^{1/2})$,

$$\begin{aligned}R^A(X,Y)(\psi\otimes\sigma)&=\big(R(X,Y)\psi\big)\otimes\sigma+\psi\otimes R^{L^{1/2},A}(X,Y)\sigma\\&=\big(R(X,Y)\psi\big)\otimes\sigma-\frac{\sqrt{-1}}{2}\Omega^A(X,Y)\psi\otimes\sigma,\end{aligned}$$

where R is the curvature of the local spinor bundle. We deduce that

$$\begin{aligned}\mathcal{R}_H^A(\psi\otimes\sigma)&=-\frac{1}{2}\sum_{i,j}\varepsilon_i.\varepsilon_j.\left(\big(R_H(\varepsilon_i,\varepsilon_j)\psi\big)\otimes\sigma-\frac{\sqrt{-1}}{2}\Omega_H^A(\varepsilon_i,\varepsilon_j)\psi\otimes\sigma\right)\\&=(\mathcal{R}_H\psi)\otimes\sigma+\frac{\sqrt{-1}}{2}(\Omega_H^A.\psi)\otimes\sigma,\end{aligned}$$

and

$$\begin{aligned}\mathcal{R}_\xi^A(\psi\otimes\sigma)&=-\sum_i\xi.\varepsilon_i.\left(\big(R(\xi,\varepsilon_i)\psi\big)\otimes\sigma-\frac{\sqrt{-1}}{2}\Omega^A(\xi,\varepsilon_i)\psi\otimes\sigma\right)\\&=(\mathcal{R}_\xi\psi)\otimes\sigma+\frac{\sqrt{-1}}{2}\big(\xi.i(\xi)\Omega^A.\psi\big)\otimes\sigma.\end{aligned}$$

Now, R is given by

$$R(X,Y)=\frac{1}{4}\sum_{1\leqslant i,j\leqslant 2d}g_\theta\big(R(X,Y)\varepsilon_i,\varepsilon_j\big)\varepsilon_i.\varepsilon_j. \tag{14}$$

Using (14), we have

$$
\begin{aligned}
\mathcal{R}_H &= -\frac{1}{8}\sum_{1\leqslant i,j,k,l\leqslant 2d} g_\theta\big(R_H(\varepsilon_i,\varepsilon_j)\varepsilon_k,\varepsilon_l\big)\varepsilon_i.\varepsilon_j.\varepsilon_k.\varepsilon_l \\
&= -\frac{1}{8}\sum_{1\leqslant l\leqslant 2d}\Bigg[\sum_{1\leqslant i\neq j\neq k\leqslant 2d} g_\theta\big(R_H(\varepsilon_i,\varepsilon_j)\varepsilon_k,\varepsilon_l\big)\varepsilon_i.\varepsilon_j.\varepsilon_k + 2\sum_{1\leqslant i,j\leqslant 2d} g_\theta\big(R_H(\varepsilon_i,\varepsilon_j)\varepsilon_i,\varepsilon_l\big)\varepsilon_j\Bigg].\varepsilon_l \\
&= -\frac{1}{24}\sum_{1\leqslant l\leqslant 2d}\Bigg[\sum_{1\leqslant i\neq j\neq k\leqslant 2d} g_\theta\big(R_H(\varepsilon_i,\varepsilon_j)\varepsilon_k + R_H(\varepsilon_k,\varepsilon_i)\varepsilon_j + R_H(\varepsilon_j,\varepsilon_k)\varepsilon_i,\varepsilon_l\big)\varepsilon_i.\varepsilon_j.\varepsilon_k\Bigg].\varepsilon_l \\
&\quad -\frac{1}{4}\sum_{1\leqslant l\leqslant 2d}\Bigg[\sum_{1\leqslant i,j\leqslant 2d} g_\theta\big(R_H(\varepsilon_i,\varepsilon_j)\varepsilon_i,\varepsilon_l\big)\varepsilon_j\Bigg].\varepsilon_l \\
&= -\frac{1}{24}\sum_{1\leqslant l\leqslant 2d}\Bigg[\sum_{1\leqslant i\neq j\neq k\leqslant 2d} g_\theta\big(\omega_\theta(\varepsilon_i,\varepsilon_j)\tau(\varepsilon_k) + \omega_\theta(\varepsilon_k,\varepsilon_i)\tau(\varepsilon_j) \\
&\quad + \omega_\theta(\varepsilon_j,\varepsilon_k)\tau(\varepsilon_i),\varepsilon_l\big)\varepsilon_i.\varepsilon_j.\varepsilon_k\Bigg].\varepsilon_l - \frac{1}{4}\sum_{1\leqslant i,j\leqslant 2d}\varepsilon_j.R_H(\varepsilon_i,\varepsilon_j)\varepsilon_i \\
&= -\frac{1}{8}\sum_{1\leqslant i,j,k,l\leqslant 2d} g_\theta\big(\omega_\theta(\varepsilon_i,\varepsilon_j)\tau(\varepsilon_k),\varepsilon_l\big)\varepsilon_i.\varepsilon_j.\varepsilon_k.\varepsilon_l + \frac{1}{4}\sum_{1\leqslant i,j,l\leqslant 2d} g_\theta\big(\omega_\theta(\varepsilon_i,\varepsilon_j)\tau(\varepsilon_i),\varepsilon_l\big)\varepsilon_j.\varepsilon_l \\
&\quad -\frac{1}{4}\sum_{1\leqslant j\leqslant 2d}\varepsilon_j.\operatorname{Ric}(\varepsilon_j) \\
&= -\frac{1}{4}\omega_\theta.\sum_{1\leqslant i\leqslant 2d}\varepsilon_i.\tau(\varepsilon_i) + \frac{1}{4}\sum_{1\leqslant i\leqslant 2d}\varepsilon_i.J\tau(\varepsilon_i) - \frac{1}{4}\sum_{1\leqslant j\leqslant 2d}\varepsilon_j.\operatorname{Ric}(\varepsilon_j).
\end{aligned}
$$

Since τ and $J\circ\tau$ are symmetric endomorphisms, then the two first terms in the last expression are zero. Restricted to H, the endomorphism Ric is symmetric. We deduce that

$$
-\frac{1}{4}\sum_{1\leqslant j\leqslant 2d}\varepsilon_j.\operatorname{Ric}(\varepsilon_j) = \frac{1}{4}\sum_{1\leqslant j\leqslant 2d} g_\theta\big(\operatorname{Ric}(\varepsilon_j),\varepsilon_j\big) = \frac{1}{4}s.
$$

Now, we have to calculate the term $\mathcal{R}_\xi$.

$$
\begin{aligned}
\mathcal{R}_\xi &= -\frac{1}{4}\sum_{1\leqslant i,j,k\leqslant 2d} g_\theta\big(R(\xi,\varepsilon_i)\varepsilon_j,\varepsilon_k\big)\xi.\varepsilon_i.\varepsilon_j.\varepsilon_k \\
&= -\frac{1}{4}\Bigg[\sum_{1\leqslant i\neq j\neq k\leqslant 2d} g_\theta\big(R(\xi,\varepsilon_i)\varepsilon_j,\varepsilon_k\big)\xi.\varepsilon_i.\varepsilon_j.\varepsilon_k - 2\sum_{1\leqslant j,k\leqslant 2d} g_\theta\big(R(\xi,\varepsilon_j)\varepsilon_j,\varepsilon_k\big)\xi.\varepsilon_k\Bigg] \\
&= -\frac{1}{12}\sum_{1\leqslant i\neq j\neq k\leqslant 2d}\big[g_\theta\big(R(\xi,\varepsilon_i)\varepsilon_j,\varepsilon_k\big) + g_\theta\big(R(\xi,\varepsilon_k)\varepsilon_i,\varepsilon_j\big) + g_\theta\big(R(\xi,\varepsilon_j)\varepsilon_k,\varepsilon_i\big)\big]\xi.\varepsilon_i.\varepsilon_j.\varepsilon_k \\
&\quad -\frac{1}{2}\sum_{1\leqslant j\leqslant 2d}\xi.R(\varepsilon_j,\xi)\varepsilon_j
\end{aligned}
$$

$$= -\frac{1}{12}\sum_{1\leqslant i\neq j\neq k\leqslant 2d}\bigl[g_\theta\bigl(R(\xi,\varepsilon_i)\varepsilon_j + R(\varepsilon_j,\xi)\varepsilon_i,\varepsilon_k\bigr) - g_\theta\bigl(R(\varepsilon_k,\xi)\varepsilon_i,\varepsilon_j\bigr)\bigr]\xi.\varepsilon_i.\varepsilon_j.\varepsilon_k$$
$$- \frac{1}{2}\xi.\operatorname{Ric}(\xi)$$
$$= -\frac{1}{12}\sum_{1\leqslant i\neq j\neq k\leqslant 2d}\bigl[g_\theta\bigl((\nabla_{\varepsilon_j}\tau)(\varepsilon_i) - (\nabla_{\varepsilon_i}\tau)(\varepsilon_j),\varepsilon_k\bigr) - g_\theta\bigl((\nabla_{\varepsilon_j}\tau)(\varepsilon_k),\varepsilon_i\bigr)$$
$$+ g_\theta\bigl((\nabla_{\varepsilon_i}\tau)(\varepsilon_k),\varepsilon_j\bigr)\bigr]\xi.\varepsilon_i.\varepsilon_j.\varepsilon_k - \frac{1}{2}\xi.\delta\tau.$$

Since $g_\theta((\nabla_X\tau)(Y),Z) = g_\theta((\nabla_X\tau)(Z),Y)$, then the first term in the last expression is zero. Hence the result. Now, suppose that M is strictly pseudoconvex (i.e., $\nabla J = \nabla\omega_\theta = 0$) and consider the 2-forms

$$Q^{\mathrm{can}}_{H_0} = Q^{\mathrm{can}}_H - \frac{1}{d}\omega_\theta\otimes(\wedge_H Q^{\mathrm{can}}_H) \quad\text{and}\quad Q^{\mathrm{can}}_{H_-} = \frac{1}{2}(Q^{\mathrm{can}}_H - Q^{\mathrm{can}}_H\circ J) = (Q^{\mathrm{can}}_{H_-})_0.$$

$Q^{\mathrm{can}}_{H_0}$ and $Q^{\mathrm{can}}_{H_-}$ are parallel and primitive. Formula (8) for $Q^{\mathrm{can}}_{H_0}$ and $Q^{\mathrm{can}}_{H_-}$ gives respectively

$$\mathcal{D}^{A^2}_H - \bigl((\nabla^A_H)^*\nabla^A_H\bigr)_0 = \mathcal{R}^A_{H_0}$$

and

$$\mathcal{D}^{A^2}_{H,J} - \mathcal{D}^{A^2}_H = -2\mathcal{R}^A_{H_-},$$

where

$$\bigl((\nabla^A_H)^*\nabla^A_H\bigr)_0 = -\sum_i\left(\nabla^{A^2}_{\varepsilon_i,\varepsilon_i} - \frac{1}{d}\omega_\theta.\nabla^{A^2}_{\varepsilon_i,J\varepsilon_i}\right)$$

and where $\mathcal{R}^A_{H_0}$ and $\mathcal{R}^A_{H_-}$ are the endomorphisms locally given by:

$$\mathcal{R}^A_{H_0} = -\frac{1}{2}\sum_{i,j}\varepsilon_i.\varepsilon_j.R^A_{H_0}(\varepsilon_i,\varepsilon_j), \qquad \mathcal{R}^A_{H_-} = -\frac{1}{2}\sum_{i,j}\varepsilon_i.\varepsilon_j.R^A_{H_-}(\varepsilon_i,\varepsilon_j).$$

Let $\{e_1, Je_1,\dots,e_d, Je_d\}$ be a local orthonormal frame of H and $Z_i = \frac{1}{2}(e_i - \sqrt{-1}Je_i)$. A direct calculation yields

$$\left(1 + \frac{\sqrt{-1}}{d}\omega_\theta\right)(\nabla^A_{0,1})^*\nabla^A_{0,1} + \left(1 - \frac{\sqrt{-1}}{d}\omega_\theta\right)(\nabla^A_{1,0})^*\nabla^A_{1,0}$$
$$= -\frac{1}{2}\sum_{i=1}^d\left(\nabla^{A^2}_{e_i,e_i} + \nabla^{A^2}_{Je_i,Je_i} - \frac{1}{d}\omega_\theta.\bigl(\nabla^{A^2}_{e_i,Je_i} - \nabla^{A^2}_{Je_i,e_i}\bigr)\right) = \frac{1}{2}\bigl((\nabla^A_H)^*\nabla^A_H\bigr)_0.$$

Now, we have

$$\mathcal{R}^A_{H_0}(\psi\otimes\sigma) = (\mathcal{R}_{H_0}\psi)\otimes\sigma + \frac{\sqrt{-1}}{2}(\Omega^A_{H_0}.\psi)\otimes\sigma$$
$$= \left(\frac{1}{4}s\psi + \frac{1}{d}\omega_\theta.(\wedge_H R_H)\psi\right)\otimes\sigma + \frac{\sqrt{-1}}{2}(\Omega^A_{H_0}.\psi)\otimes\sigma,$$

and

$$\mathcal{R}^A_{H_-}(\psi\otimes\sigma)=(\mathcal{R}_{H_-}\psi)\otimes\sigma+\frac{\sqrt{-1}}{2}(\Omega^A_{H_-}.\psi)\otimes\sigma.$$

We have

$$\wedge_H R_H=\frac{1}{8}\sum_{1\leqslant i,j,k\leqslant 2d} g_\theta\big(R_H(\varepsilon_i,J\varepsilon_i)\varepsilon_j,\varepsilon_k\big)\varepsilon_j.\varepsilon_k=\frac{1}{4}\sum_{1\leqslant j,k\leqslant 2d} g_\theta\big((\wedge_H R_H)(\varepsilon_j),\varepsilon_k\big)\varepsilon_j.\varepsilon_k.$$

For any $X\in H$, we have using (1),

$$R_H(\varepsilon_i,J\varepsilon_i)X+R_H(X,\varepsilon_i)J\varepsilon_i+R_H(J\varepsilon_i,X)\varepsilon_i=\tau(X)+\omega_\theta(X,\varepsilon_i)\tau(J\varepsilon_i)-g_\theta(X,\varepsilon_i)\tau(\varepsilon_i).$$

Hence, we obtain

$$(\wedge_H R_H)(X)-J\,\mathrm{Ric}(X)=(d-1)\tau(X).$$

Using (2), we obtain that, for $Y\in H$

$$g_\theta\big((\wedge_H R_H)(X),Y\big)=-g_\theta\big(\mathrm{Ric}(X),JY\big)+(d-1)g_\theta\big(\tau(X),Y\big)=-\mathrm{Ric}_{H_+}(X,JY)=-\rho_H(X,Y).$$

We deduce that $\wedge_H R_H=-\frac{1}{2}\rho_H$.

Using (3), we have

$$\begin{aligned}
\mathcal{R}_{H_-}&=-\frac{1}{8}\sum_{1\leqslant i,j,k,l\leqslant 2d} g_\theta\big(R_{H_-}(\varepsilon_i,\varepsilon_j)\varepsilon_k,\varepsilon_l\big)\varepsilon_i.\varepsilon_j.\varepsilon_k.\varepsilon_l\\
&=-\frac{1}{24}\sum_{1\leqslant l\leqslant 2d}\Bigg[\sum_{1\leqslant i\neq j\neq k\leqslant 2d} g_\theta\big(R_{H_-}(\varepsilon_i,\varepsilon_j)\varepsilon_k+R_{H_-}(\varepsilon_k,\varepsilon_i)\varepsilon_j+R_{H_-}(\varepsilon_j,\varepsilon_k)\varepsilon_i,\varepsilon_l\big)\varepsilon_i.\varepsilon_j.\varepsilon_k\Bigg].\varepsilon_l\\
&\quad-\frac{1}{4}\sum_{1\leqslant l\leqslant 2d}\Bigg[\sum_{1\leqslant i,j\leqslant 2d} g_\theta\big(R_{H_-}(\varepsilon_i,\varepsilon_j)\varepsilon_i,\varepsilon_l\big)\varepsilon_j\Bigg].\varepsilon_l\\
&=\frac{1}{24}\sum_{1\leqslant l\leqslant 2d}\Bigg[\sum_{1\leqslant i\neq j\neq k\leqslant 2d} g_\theta\big(\omega_\theta(\varepsilon_i,\varepsilon_j)\tau(\varepsilon_k)+\omega_\theta(\varepsilon_k,\varepsilon_i)\tau(\varepsilon_j)\\
&\quad+\omega_\theta(\varepsilon_j,\varepsilon_k)\tau(\varepsilon_i),\varepsilon_l\big)\varepsilon_i.\varepsilon_j.\varepsilon_k\Bigg].\varepsilon_l-\frac{1}{4}\sum_{1\leqslant l\leqslant 2d}\Bigg[\sum_{1\leqslant i,j\leqslant 2d} g_\theta(R_{H_-}(\varepsilon_i,\varepsilon_j)\varepsilon_i,\varepsilon_l)\varepsilon_j\Bigg].\varepsilon_l\\
&=-\frac{1}{4}\sum_{1\leqslant l\leqslant 2d}\Bigg[\sum_{1\leqslant i,j\leqslant 2d} g_\theta\big(R_{H_-}(\varepsilon_i,\varepsilon_j)\varepsilon_i,\varepsilon_l\big)\varepsilon_j\Bigg].\varepsilon_l.
\end{aligned}$$

Now, for $X,Y\in H$, we have using (2)

$$\sum_{1\leqslant i\leqslant 2d} g_\theta\big(R_{H_-}(\varepsilon_i,X)\varepsilon_i,Y\big)=\mathrm{Ric}_{H_-}(X,Y)=(d-1)\omega_\theta\big(\tau(X),Y\big). \tag{15}$$

Formula (15) yields

$$\mathcal{R}_{H_-}=-\frac{d-1}{4}\Bigg(\sum_{1\leqslant j\leqslant 2d}\varepsilon_j.J\tau(\varepsilon_j)\Bigg)=0. \qquad\square$$

4.2. Vanishing theorems

Definition 4.1. A spinor field $\psi \in \Gamma(\Sigma M^c)$ such that $\mathcal{D}_A\psi = 0$ (resp. $\mathcal{D}_H^A\psi = 0$) will be called a harmonic spinor (resp. subharmonic spinor).

Definition 4.2 [2]. A contact metric manifold for which the Tanaka–Webster torsion vanishes is called a k-contact metric manifold.

Theorem 4.1. *Let M be a compact k-contact metric manifold of dimension $m \geqslant 3$ endowed with a* Spin^c*-structure. We suppose that the determinant line bundle L is endowed with a unitary connection A for which the curvature form Ω^A is horizontal. If $\frac{1}{2}s \geqslant |\lambda_1| + \cdots + |\lambda_d|$ at each point, where $\lambda_1, \ldots, \lambda_d$ are the eigenvalues of Ω^A, then any harmonic spinor is parallel. If the previous inequality is strict at some point, then there is no harmonic spinor.*

Proof. Since M has Tanaka–Webster torsion vanishing and that $i(\xi)\Omega^A = 0$, Formula (10) yields $\{\mathcal{D}_H^A, \lambda_\xi \circ \nabla_\xi^A\} = 0$. We deduce by (6), (7) that

$$\mathcal{D}_A^2 = \mathcal{D}_H^{A^2} - \nabla_{\xi,\xi}^{A^2}.$$

Let $\psi \in \Gamma(\Sigma M^c)$, we obtain by integrating

$$\int_M |\mathcal{D}_A\psi|^2 v_{g_\theta} = \int_M |\mathcal{D}_H^A\psi|^2 + |\nabla_\xi^A\psi|^2 v_{g_\theta}.$$

If ψ is a harmonic spinor, then we deduce from the last equation that $\mathcal{D}_H^A\psi = \nabla_\xi^A\psi = 0$. Hence, we have using (11):

$$\int_M |\nabla_H^A\psi|^2 + \frac{1}{4}s|\psi|^2 + \frac{\sqrt{-1}}{2}\langle\Omega_H^A.\psi, \psi\rangle v_{g_\theta} = 0. \tag{16}$$

Now, we can find a local orthonormal frame $\{e_i\}$ of H such that

$$\Omega_H^A = \sum_{1\leqslant i\leqslant d} \lambda_i e_{2i-1} \wedge e_{2i}.$$

We deduce that $|\langle\Omega_H^A.\psi, \psi\rangle| \leqslant |\Omega_H^A.\psi||\psi| \leqslant (\sum_{1\leqslant i\leqslant d}|\lambda_i|)|\psi|^2$ and, using the assumption on s, that $\frac{1}{4}s|\psi|^2 + \frac{\sqrt{-1}}{2}\langle\Omega_H^A.\psi, \psi\rangle \geqslant 0$. The result follows immediately from (16). $\square$

If M is spin, then L is trivial and flat. We deduce the following:

Theorem 4.2. *Let M be a compact spin k-contact metric manifold of dimension $m \geqslant 3$. Suppose that the Tanaka–Webster scalar curvature of M is nonnegative and positive at some point, then there is no harmonic spinor.*

In the following of this section we suppose that M is a strictly pseudoconvex CR manifold.

Definition 4.3. A strictly pseudoconvex *CR* manifold for which the Tanaka–Webster torsion vanishes is called a Sasakian manifold.

Corollary 4.1. *Let M be a compact Sasakian manifold of dimension $m \geqslant 3$ with pseudo-Hermitian Ricci tensor nonnegative. Then any harmonic spinor for the canonical* Spin^c*-structure is parallel.*

Proof. Let A_{ac} be the Webster connection on K^{-1}, then we have (cf. [1]):

$$\Omega_H^{A_{\mathrm{ac}}}(X,Y) = -\frac{1}{2}\big(\omega_\theta\big(X, \mathrm{Ric}(Y)\big) + \omega_\theta\big(\mathrm{Ric}(X), Y\big)\big) = \rho_H(X,Y).$$

Moreover, M is a Sasakian manifold, hence we have $\Omega^{A_{\mathrm{ac}}}(\xi, X) = 0$. Now, we can find a local orthonormal frame $\{\varepsilon_1, J\varepsilon_1, \dots, \varepsilon_d, J\varepsilon_d\}$ of H such that

$$\rho_H = \sum_{1\leqslant i\leqslant d} \lambda_i \varepsilon_i \wedge J\varepsilon_i,$$

with $\lambda_i = -\mathrm{Ric}_{H_+}(\varepsilon_i, \varepsilon_i)$. Since the pseudo-Hermitian Ricci tensor is nonnegative, we deduce that $|\lambda_1| + \cdots + |\lambda_d| = \frac{1}{2}s$. The result follows from Theorem 4.1. □

Remember [17,19] that a Hermitian vector bundle E over a strictly pseudoconvex *CR* manifold M endowed with a Hermitian connection ∇ is holomorphic if and only if $R_H^{(0,2)} = 0$ where R is the curvature of ∇. A section σ of a holomorphic bundle E is said to be holomorphic if $(\bar\partial_H^E \sigma)(\overline{Z}) = \nabla_{\overline{Z}}\sigma = 0$ for any $\overline{Z} \in T^{0,1}M$. Note that the canonical and anticanonical bundles defined above are examples of holomorphic (line) bundles.

Theorem 4.3. *Let M be a compact strictly pseudoconvex CR manifold of dimension $m \geqslant 3$ with pseudo-Hermitian Ricci tensor nonnegative and pseudo-Hermitian scalar curvature positive at some point. We suppose M endowed with a* Spin^c*-structure for which the determinant line bundle L is endowed with a holomorphic, unitary connection A such that $(\Omega_{H_0}^A)_+ = 0$. Then, for any subharmonic spinor ψ, we have $\psi = \psi_1 + \psi_2$, where ψ_1 is a holomorphic section of $\mathcal{L}$ and $\overline{\psi_2}$ is a holomorphic section of $K \otimes \mathcal{L}^{-1}$.*

Proof. Since A is a holomorphic connection on L and $(\Omega_{H_0}^A)_+ = 0$, we have $\Omega_{H_0}^A = 0$. Let ψ be a subharmonic spinor on M, we obtain by integrating equation (12):

$$\int_M \left\langle \left(1 + \frac{\sqrt{-1}}{d}\omega_\theta\right).\nabla_{0,1}^A\psi, \nabla_{0,1}^A\psi \right\rangle + \left\langle \left(1 - \frac{\sqrt{-1}}{d}\omega_\theta\right).\nabla_{1,0}^A\psi, \nabla_{1,0}^A\psi \right\rangle$$
$$+ \frac{1}{8}\left(s|\psi|^2 - \frac{2}{d}\langle \omega_\theta.\rho_H.\psi, \psi\rangle\right) v_{g_\theta} = 0. \tag{17}$$

Now, let $\{\varepsilon_1, J\varepsilon_1, \dots, \varepsilon_d, J\varepsilon_d\}$ be a local orthonormal frame of H such that

$$\omega_\theta = \sum_{1\leqslant i\leqslant d} \varepsilon_i \wedge J\varepsilon_i \quad \text{and} \quad \rho_H = \sum_{1\leqslant i\leqslant d} \lambda_i \varepsilon_i \wedge J\varepsilon_i,$$

with $\lambda_i = -\mathrm{Ric}_{H_+}(\varepsilon_i, \varepsilon_i)$. If the pseudo-Hermitian Ricci tensor is nonnegative, then $\lambda_i \leqslant 0$ for $i \in \{1, \dots, d\}$, and we obtain that $|\rho_H.\psi| \leqslant (\sum_{1\leqslant i\leqslant d} |\lambda_i|)|\psi| = \frac{s}{2}|\psi|$. Using the inequality $|\omega_\theta.\psi| \leqslant d|\psi|$, we have

$$\left|\langle\omega_\theta.\rho_H.\psi,\psi\rangle\right| = \left|\langle\rho_H.\psi,\omega_\theta.\psi\rangle\right| \leqslant |\rho_H.\psi||\omega_\theta.\psi| \leqslant \frac{sd}{2}|\psi|^2. \tag{18}$$

Since $\langle\omega_\theta.\rho_H.\psi,\psi\rangle \in \mathbb{R}$, we deduce that $s|\psi|^2 - \frac{2}{d}\langle\omega_\theta.\rho_H.\psi,\psi\rangle \geqslant 0$. Now, we have $\psi = \sum_{q=0}^{d} \psi^{d-2q}$ with $\psi^{d-2q} \in \Sigma_{(d-2q)}M^c \simeq \wedge_H^{0,q}(M)\otimes\mathcal{L}$. We deduce that

$$\left(1+\frac{\sqrt{-1}}{d}\omega_\theta\right)\psi = 2\sum_{q=0}^{d}\left(1-\frac{q}{d}\right)\psi^{d-2q} \quad \text{and} \quad \left(1-\frac{\sqrt{-1}}{d}\omega_\theta\right)\psi = 2\sum_{q=0}^{d}\frac{q}{d}\psi^{d-2q}.$$

Hence (17) becomes

$$\int_M \sum_{q=0}^{d}\left(\left(1-\frac{q}{d}\right)\left|\nabla_{0,1}^{A}(\psi^{d-2q})\right|^2 + \frac{q}{d}\left|\nabla_{1,0}^{A}(\psi^{d-2q})\right|^2\right) + \frac{1}{16}\left(s|\psi|^2 - \frac{2}{d}\langle\omega_\theta.\rho_H.\psi,\psi\rangle\right)v_{g_\theta} = 0. \tag{19}$$

Since $s|\psi|^2 - \frac{2}{d}\langle\omega_\theta.\rho_H.\psi,\psi\rangle \geqslant 0$, we obtain from (19) that, $\nabla_H^A\psi^{d-2q} = 0$ for any $q \in \{1,\dots,d-1\}$, $\nabla_{\bar{Z}}^A(\psi^d) = 0$ and $\nabla_Z^A(\psi^{-d}) = 0$, for any $Z \in T^{1,0}M$. Since $\mathcal{L}$ is holomorphic, the two last equations imply that ψ^d is a holomorphic section of $\Sigma_d M^c \simeq \mathcal{L}$ and $\overline{\psi^{-d}}$ is a holomorphic section of $\overline{\Sigma_{(-d)}M^c} \simeq K\otimes\mathcal{L}^{-1}$. Now (19) also gives $\langle\omega_\theta.\rho_H.\psi,\psi\rangle = \frac{sd}{2}|\psi|^2$. Consequently, the Schwarz inequality in (18) is an equality and we obtain that $\rho_H.\psi = f\omega_\theta.\psi$, $f \in C^\infty(M)$. We deduce that $f|\omega_\theta.\psi|^2 = -\frac{sd}{2}|\psi|^2$. Since $|\rho_H.\psi||\omega_\theta.\psi| = \frac{sd}{2}|\psi|^2$ and $|\rho_H.\psi| \leqslant \frac{s}{2}|\psi|$, we have $|\omega_\theta.\psi| \geqslant d|\psi|$. Hence $|\omega_\theta.\psi| = d|\psi|$ and $f = -\frac{s}{2d}$. Now, we have $|\omega_\theta.\psi|^2 = \sum_{q=0}^{d}(d-2q)^2|\psi^{d-2q}|^2$ and

$$-\frac{s}{2d}|\omega_\theta.\psi|^2 + \frac{sd}{2}|\psi|^2 = \sum_{q=0}^{d}\frac{sd}{2}\left(1-\left(1-\frac{2q}{d}\right)^2\right)|\psi^{d-2q}|^2 = 2s\sum_{q=0}^{d}q\left(1-\frac{q}{d}\right)|\psi^{d-2q}|^2 = 0.$$

At a point where $s > 0$, we deduce that $\psi^{d-2q} = 0$ for any $q \in \{1,\dots,d-1\}$. Now, for any $q \in \{1,\dots,d-1\}$, we have $\nabla_H^A\psi^{d-2q} = 0$, so $|\psi^{d-2q}|$ is constant on M. Consequently, for any $q \in \{1,\dots,d-1\}$, $\psi^{d-2q} = 0$ everywhere on M and $\psi = \psi^d + \psi^{-d}$. □

Corollary 4.2. *Let M be a compact spin strictly pseudoconvex CR manifold of dimension $m \geqslant 3$ with pseudo-Hermitian Ricci tensor nonnegative and pseudo-Hermitian scalar curvature positive at some point. Then, for any subharmonic spinor ψ, we have $\psi = \psi_1 + \psi_2$, where ψ_1 and $\overline{\psi_2}$ are holomorphic sections of $K^{1/2}$.*

Definition 4.4. A strictly pseudoconvex *CR* manifold for which $\rho_H = f\omega_\theta$, $f \in C^\infty(M)$, is called pseudo-Einstein ($f = -\frac{s}{2d}$).

Recall that the condition pseudo-Einstein does not imply that the pseudo-Hermitian scalar curvature is constant. If M is pseudo-Einstein, then $\Omega_{H_0}^{A_{\mathrm{ac}}} = 0$. We deduce the following corollary:

Corollary 4.3. *Let M be a compact strictly pseudoconvex CR manifold of dimension $m \geqslant 3$ endowed with its canonical* Spin^c*-structure. Suppose that M is pseudo-Einstein with pseudo-Hermitian scalar curvature nonnegative and positive at some point. Then, for any subharmonic spinor ψ, we have $\psi = \psi_1 + \psi_2$, where ψ_1 is a* (*CR-*) *holomorphic function on M and $\overline{\psi_2}$ is a holomorphic section of K.*

5. Changes of metrics on a spin contact metric manifold and Dirac operator

Following the works of Bourguignon–Gauduchon [3] and Maier [8] concerning the infinitesimal variation of the Dirac operator with respect to variations of metrics, we are interesting, in this section, to the infinitesimal variation of the Kohn–Dirac operator on a spin contact manifold with respect to variations of a metric associated to a contact form.

Let (M, θ, ξ) be a contact manifold and let $\mathcal{M}(\theta)$ be the set of metrics associated to θ. For $g_\theta, h_\theta \in \mathcal{M}(\theta)$, we consider G_{h_θ, g_θ} the g_θ-symmetric endomorphism of $T_x M$ given by $h_\theta(X, Y) = g_\theta(G_{h_\theta, g_\theta}(X), Y)$. Note that G_{h_θ, g_θ} is positive and that $G_{h_\theta, g_\theta}(\xi) = \xi$. Now, let $b_{h_\theta, g_\theta} = G_{h_\theta, g_\theta}^{-1/2}$, then b_{h_θ, g_θ} is a smooth SO_{2d+1}-equivariant map from $P_{SO_{2d+1}}(M, g_\theta)$ to $P_{SO_{2d+1}}(M, h_\theta)$. If M is spin, then there exists a smooth $Spin_{2d+1}$-equivariant map $\beta_{h_\theta, g_\theta} : P_{Spin_{2d+1}}(M, g_\theta) \to P_{Spin_{2d+1}}(M, h_\theta)$ which covers b_{h_θ, g_θ}. Let $\Sigma_{g_\theta} M$ and $\Sigma_{h_\theta} M$ be the associated spinor bundles, then $\beta_{h_\theta, g_\theta}$ extends to an isometry $\beta_{h_\theta, g_\theta} : \Sigma_{g_\theta} M \to \Sigma_{h_\theta} M$ and we have

$$\beta_{h_\theta, g_\theta}(X.\psi) = b_{h_\theta, g_\theta}(X).\beta_{h_\theta, g_\theta}(\psi).$$

We denote by ∇^{h_θ} (resp. ∇^{g_θ}) the Tanaka–Webster connection for h_θ (resp. g_θ) on TM and $\Sigma_{h_\theta} M$ (resp. $\Sigma_{g_\theta} M$). The connection on $\Sigma_{g_\theta} M$ given by $\beta_{h_\theta, g_\theta}^{-1} \circ \nabla^{h_\theta} \circ \beta_{h_\theta, g_\theta}$ (induced by the g_θ-metric connection $b_{h_\theta, g_\theta}^{-1} \circ \nabla^{h_\theta} \circ b_{h_\theta, g_\theta}$ on TM) will be denoted by $\nabla^{h_\theta, g_\theta}$. Now, let $\mathcal{D}_H^{h_\theta, g_\theta}$ the differential operator on $\Gamma(\Sigma_{g_\theta} M)$ defined by

$$\mathcal{D}_H^{h_\theta, g_\theta} = \beta_{h_\theta, g_\theta}^{-1} \circ \mathcal{D}_H^{h_\theta} \circ \beta_{h_\theta, g_\theta}.$$

We have, for any $\psi \in \Gamma(\Sigma_{g_\theta} M)$,

$$\begin{aligned}
\mathcal{D}_H^{h_\theta, g_\theta} \psi &= \sum_i \varepsilon_i . \nabla^{h_\theta, g_\theta}_{b_{h_\theta, g_\theta}(\varepsilon_i)} \psi \\
&= \frac{1}{2} \sum_i \varepsilon_i . \big(b_{h_\theta, g_\theta}^{-1} \circ \nabla^{g_\theta}_{b_{h_\theta, g_\theta}(\varepsilon_i)} b_{h_\theta, g_\theta} + b_{h_\theta, g_\theta}^{-1} \circ A^{h_\theta, g_\theta}\big(b_{h_\theta, g_\theta}(\varepsilon_i)\big) \circ b_{h_\theta, g_\theta} \big) . \psi \\
&\quad + \sum_i \varepsilon_i . \nabla^{g_\theta}_{b_{h_\theta, g_\theta}(\varepsilon_i)} \psi,
\end{aligned}$$

where, $\{\varepsilon_i\}_{i \leqslant 2d}$ is a local g_θ-orthonormal frame of H, . the Clifford multiplication for g_θ, and $A^{h_\theta, g_\theta}(X) = \nabla^{h_\theta}_X - \nabla^{g_\theta}_X$.

Let $(M, \theta, \xi, J, g_\theta)$ be a contact metric manifold and let τ be the Tanaka–Webster torsion of ∇. For any $t \in \mathbb{R}$, we define

$$g_\theta^t(X, Y) = g_\theta\big(e^{t\tau}(X), Y\big) \quad \text{and} \quad J_t = J \circ e^{t\tau}.$$

It follows directly from Proposition 4.1 of [18] that $g_\theta^t \in \mathcal{M}(\theta)$ and so $(M, \theta, \xi, J_t, g_\theta^t)$ is a contact metric manifold. Note that $\upsilon_{g_\theta^t} = \upsilon_{g_\theta}$.

Lemma 5.1. *Let ∇^t be the Tanaka–Webster connection of g_θ^t and let τ^t be the Tanaka–Webster torsion of ∇^t, then we have*

$$\tau^t = \tau - \frac{1}{2}(e^{-t\tau} \circ \nabla_\xi e^{t\tau}).$$

Proof. We have

$$\tau^t(X) = \frac{1}{2}(J_t \circ \mathcal{L}_\xi J_t)(X) = \frac{1}{2}\big(Je^{t\tau}\big(\big[\xi, Je^{t\tau}(X)\big] - Je^{t\tau}[\xi, X]\big)\big).$$

Since $J \circ e^{t\tau} = e^{-t\tau} \circ J$, we have

$$\tau^t(X) = \frac{1}{2}\big([\xi, X] + Je^{t\tau}\big(\big[\xi, e^{-t\tau}(JX)\big]\big)\big).$$

Using the relation $\tau(X) = [\xi, X] - \nabla_\xi X$, we obtain that

$$\big[\xi, e^{-t\tau}(X)\big] = e^{-t\tau}\big([\xi, X]\big) + (\nabla_\xi e^{-t\tau})(X).$$

Now, since $(\nabla_\xi e^{-t\tau})(JX) = J(\nabla_\xi e^{t\tau})(X)$, we deduce that

$$\big[\xi, e^{-t\tau}(JX)\big] = e^{-t\tau}\big([\xi, JX]\big) + J(\nabla_\xi e^{t\tau})(X).$$

It follows that

$$\begin{aligned}
\tau^t(X) &= \frac{1}{2}\big([\xi, X] + J[\xi, JX] + Je^{t\tau}\big(J(\nabla_\xi e^{t\tau})(X)\big)\big)\\
&= \frac{1}{2}\big(J\big([\xi, JX] - J[\xi, X]\big) - e^{-t\tau}\big((\nabla_\xi e^{t\tau})(X)\big)\big)\\
&= \tau(X) - \frac{1}{2}(e^{-t\tau} \circ \nabla_\xi e^{t\tau})(X). \qquad \square
\end{aligned}$$

Lemma 5.2. *Let $A_t(X) = \nabla_X^t - \nabla_X$, then we have*

$$g_\theta^t\big(A_t(X)Y, Z\big) = \frac{1}{2}\big(g_\theta\big((\nabla_X e^{t\tau})(Y), Z\big) + g_\theta\big((\nabla_{Y_H} e^{t\tau})(X), Z\big) - g_\theta\big((\nabla_{Z_H} e^{t\tau})(X), Y\big)\big). \tag{20}$$

Proof. We have

$$\begin{aligned}
(\nabla_X^t g_\theta^t)(Y, Z) &= Xg_\theta^t(Y, Z) - g_\theta^t(\nabla_X^t Y, Z) - g_\theta^t(Y, \nabla_X^t Z)\\
&= Xg_\theta^t(Y, Z) - g_\theta^t(\nabla_X Y, Z) - g_\theta^t(Y, \nabla_X Z) - g_\theta^t\big(A_t(X)Y, Z\big) - g_\theta^t\big(Y, A_t(X)Z\big)\\
&= (\nabla_X g_\theta^t)(Y, Z) - g_\theta^t\big(A_t(X)Y, Z\big) - g_\theta^t\big(Y, A_t(X)Z\big).
\end{aligned}$$

Since ∇^t is g_θ^t-metric, we deduce that

$$g_\theta^t\big(A_t(X)Y, Z\big) + g_\theta^t\big(Y, A_t(X)Z\big) = (\nabla_X g_\theta^t)(Y, Z).$$

By permutations, we obtain

$$\begin{aligned}
&g_\theta^t\big(A_t(X)Y + A_t(Y)X, Z\big) + g_\theta^t\big(A_t(X)Z - A_t(Z)X, Y\big) + g_\theta^t\big(A_t(Y)Z - A_t(Z)Y, X\big)\\
&\quad = (\nabla_X g_\theta^t)(Y, Z) + (\nabla_Y g_\theta^t)(X, Z) - (\nabla_Z g_\theta^t)(X, Y).
\end{aligned} \tag{21}$$

Now, we have $A_t(Y)X - A_t(X)Y = \nabla^t_Y X - \nabla^t_X Y - (\nabla_Y X - \nabla_X Y) = T^t(X,Y) - T(X,Y)$. Using Lemma 5.1, we obtain

$$T^t = -d\theta \otimes \xi + \theta \wedge \tau^t = -d\theta \otimes \xi + \theta \wedge \tau - \frac{1}{2}\theta \wedge (e^{-t\tau} \circ \nabla_\xi e^{t\tau}) = T - \frac{1}{2}\theta \wedge (e^{-t\tau} \circ \nabla_\xi e^{t\tau}).$$

We deduce that $A_t(Y)X - A_t(X)Y = -\frac{1}{2}(\theta \wedge (e^{-t\tau} \circ \nabla_\xi e^{t\tau}))(X,Y)$. Hence (21) becomes

$$\begin{aligned}
2g^t_\theta\big(A_t(X)Y, Z\big) &= (\nabla_X g^t_\theta)(Y,Z) + (\nabla_Y g^t_\theta)(X,Z) - (\nabla_Z g^t_\theta)(X,Y) \\
&\quad - \frac{1}{2} g^t_\theta\big(\big(\theta \wedge (e^{-t\tau} \circ \nabla_\xi e^{t\tau})\big)(Y,Z), X\big) - \frac{1}{2} g^t_\theta\big(\big(\theta \wedge (e^{-t\tau} \circ \nabla_\xi e^{t\tau})\big)(X,Z), Y\big) \\
&\quad + \frac{1}{2} g^t_\theta\big(\big(\theta \wedge (e^{-t\tau} \circ \nabla_\xi e^{t\tau})\big)(X,Y), Z\big) \\
&= (\nabla_X g^t_\theta)(Y,Z) + (\nabla_Y g^t_\theta)(X,Z) - (\nabla_Z g^t_\theta)(X,Y) \\
&\quad + \frac{1}{2}\theta(X)\big(g_\theta\big((\nabla_\xi e^{t\tau})(Y), Z\big) - g_\theta\big((\nabla_\xi e^{t\tau})(Z), Y\big)\big) \\
&\quad + \frac{1}{2}\theta(Z)\big(g_\theta\big((\nabla_\xi e^{t\tau})(Y), X\big) + g_\theta\big((\nabla_\xi e^{t\tau})(X), Y\big)\big) \\
&\quad - \frac{1}{2}\theta(Y)\big(g_\theta\big((\nabla_\xi e^{t\tau})(X), Z\big) + g_\theta\big((\nabla_\xi e^{t\tau})(Z), X\big)\big).
\end{aligned}$$

Since $(\nabla_X g^t_\theta)(Y,Z) = g_\theta((\nabla_X e^{t\tau})(Y), Z)$ and $g_\theta((\nabla_\xi e^{t\tau})(X), Y) = g_\theta((\nabla_\xi e^{t\tau})(Y), X)$ we obtain

$$\begin{aligned}
2g^t_\theta\big(A_t(X)Y, Z\big) &= g_\theta\big((\nabla_X e^{t\tau})(Y), Z\big) + g_\theta\big((\nabla_Y e^{t\tau})(X), Z\big) - g_\theta\big((\nabla_Z e^{t\tau})(X), Y\big) \\
&\quad + \theta(Z) g_\theta\big((\nabla_\xi e^{t\tau})(X), Y\big) - \theta(Y) g_\theta\big((\nabla_\xi e^{t\tau})(X), Z\big) \\
&= g_\theta\big((\nabla_X e^{t\tau})(Y), Z\big) + g_\theta\big((\nabla_{(Y-\theta(Y)\xi)} e^{t\tau})(X), Z\big) \\
&\quad - g_\theta\big((\nabla_{(Z-\theta(Z)\xi)} e^{t\tau})(X), Y\big). \qquad \square
\end{aligned}$$

Proposition 5.1. *Let $(M,\theta,\xi,J,g_\theta)$ be a spin contact metric manifold. We have for any $\psi \in \Gamma(\Sigma_{g_\theta} M)$*

$$\left(\frac{d}{dt}\big(\mathcal{D}^{g^t_\theta, g_\theta}_H\big)_{/t=0}\right)\psi = -\frac{1}{2}\left(\mathcal{D}_{H,\tau}\psi - \frac{1}{2}\delta\tau.\psi\right) = \frac{1}{2}\big(\lambda_\xi \circ \{\mathcal{D}_H, \lambda_\xi \circ \mathcal{L}_\xi\}\big)\psi.$$

Remark 5.1. This allows to consider formula (10) of Proposition 4.2 from a variational point of view.

Proof of Proposition 5.1. We have $G_{g^t_\theta, g_\theta} = e^{t\tau}$. Hence, $b_{g^t_\theta, g_\theta} = e^{-\frac{1}{2}t\tau}$, and we have

$$\begin{aligned}
\mathcal{D}^{g^t_\theta, g_\theta}_H &= \frac{1}{2}\sum_i \varepsilon_i.\big(e^{\frac{1}{2}t\tau} \circ \nabla_{e^{-\frac{1}{2}t\tau}(\varepsilon_i)} e^{-\frac{1}{2}t\tau} + e^{\frac{1}{2}t\tau} \circ A_t\big(e^{-\frac{1}{2}t\tau}(\varepsilon_i)\big) \circ e^{-\frac{1}{2}t\tau}\big) + \sum_i \varepsilon_i.\nabla_{e^{-\frac{1}{2}t\tau}(\varepsilon_i)} \\
&= \frac{1}{4}\sum_{i,j,k} g_\theta\big(\big(e^{\frac{1}{2}t\tau} \circ \nabla_{e^{-\frac{1}{2}t\tau}(\varepsilon_i)} e^{-\frac{1}{2}t\tau} + e^{\frac{1}{2}t\tau} \circ A_t\big(e^{-\frac{1}{2}t\tau}(\varepsilon_i)\big) \circ e^{-\frac{1}{2}t\tau}\big)(\varepsilon_j), \varepsilon_k\big)\varepsilon_i.\varepsilon_j.\varepsilon_k \\
&\quad + \sum_i \varepsilon_i.\nabla_{e^{-\frac{1}{2}t\tau}(\varepsilon_i)}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{4}\sum_{i,j,k}\Big(g_\theta\Big(\big(\nabla_{e^{-\frac{1}{2}t\tau}(\varepsilon_i)} e^{-\frac{1}{2}t\tau}\big)(\varepsilon_j), e^{\frac{1}{2}t\tau}(\varepsilon_k)\Big) + g^t_\theta\Big(A_t\big(e^{-\frac{1}{2}t\tau}(\varepsilon_i)\big)e^{-\frac{1}{2}t\tau}(\varepsilon_j), e^{-\frac{1}{2}t\tau}(\varepsilon_k)\Big)\Big)\varepsilon_i.\varepsilon_j.\varepsilon_k \\
&\quad + \sum_i \varepsilon_i.\nabla_{e^{-\frac{1}{2}t\tau}(\varepsilon_i)}.
\end{aligned}
$$

Using (20), we obtain

$$
\begin{aligned}
\frac{d}{dt}\big(\mathcal{D}_H^{g^t_\theta,g_\theta}\big)_{/t=0} &= \frac{1}{4}\sum_{i,j,k}\Bigg(g_\theta\Bigg(\Bigg(\nabla_{\varepsilon_i}\frac{d}{dt}\big(e^{-\frac{1}{2}t\tau}\big)_{/t=0}\Bigg)(\varepsilon_j), \varepsilon_k\Bigg) + \frac{1}{2}g_\theta\Bigg(\Bigg(\nabla_{\varepsilon_i}\frac{d}{dt}\big(e^{t\tau}\big)_{/t=0}\Bigg)(\varepsilon_j), \varepsilon_k\Bigg) \\
&\quad + \frac{1}{2}g_\theta\Bigg(\Bigg(\nabla_{\varepsilon_j}\frac{d}{dt}\big(e^{t\tau}\big)_{/t=0}\Bigg)(\varepsilon_i), \varepsilon_k\Bigg) - \frac{1}{2}g_\theta\Bigg(\Bigg(\nabla_{\varepsilon_k}\frac{d}{dt}\big(e^{t\tau}\big)_{/t=0}\Bigg)(\varepsilon_i), \varepsilon_j\Bigg)\Bigg)\varepsilon_i.\varepsilon_j.\varepsilon_k \\
&\quad + \sum_i \varepsilon_i.\nabla_{\frac{d}{dt}(e^{-\frac{1}{2}t\tau}(\varepsilon_i))_{/t=0}} \\
&= \frac{1}{8}\sum_{i,j,k}\Big(-g_\theta\big((\nabla_{\varepsilon_i}\tau)(\varepsilon_j), \varepsilon_k\big) + g_\theta\big((\nabla_{\varepsilon_i}\tau)(\varepsilon_j), \varepsilon_k\big) + g_\theta\big((\nabla_{\varepsilon_j}\tau)(\varepsilon_i), \varepsilon_k\big) \\
&\quad - g_\theta\big((\nabla_{\varepsilon_k}\tau)(\varepsilon_i), \varepsilon_j\big)\Big)\varepsilon_i.\varepsilon_j.\varepsilon_k - \frac{1}{2}\sum_i \varepsilon_i.\nabla_{\tau(\varepsilon_i)} \\
&= \frac{1}{8}\sum_{i,j,k} g_\theta\big((\nabla_{\varepsilon_k}\tau)(\varepsilon_i), \varepsilon_j\big)\varepsilon_i.(\varepsilon_k.\varepsilon_j - \varepsilon_j.\varepsilon_k) - \frac{1}{2}\sum_i \varepsilon_i.\nabla_{\tau(\varepsilon_i)} \\
&= -\frac{1}{4}\sum_{i,j,k} g_\theta\big((\nabla_{\varepsilon_k}\tau)(\varepsilon_i), \varepsilon_j\big)\varepsilon_i.\varepsilon_j.\varepsilon_k - \frac{1}{4}\sum_{i,j} g_\theta\big((\nabla_{\varepsilon_j}\tau)(\varepsilon_i), \varepsilon_j\big)\varepsilon_i - \frac{1}{2}\sum_i \varepsilon_i.\nabla_{\tau(\varepsilon_i)} \\
&= \frac{1}{4}\sum_{i,k} g_\theta\big((\nabla_{\varepsilon_k}\tau)(\varepsilon_i), \varepsilon_i\big)\varepsilon_k - \frac{1}{4}\sum_{i,j} g_\theta\big((\nabla_{\varepsilon_j}\tau)(\varepsilon_j), \varepsilon_i\big)\varepsilon_i - \frac{1}{2}\sum_i \varepsilon_i.\nabla_{\tau(\varepsilon_i)} \\
&= \frac{1}{4}\big(d_H(\mathrm{trace}_{g_\theta}\tau) + \delta\tau\big) - \frac{1}{2}\mathcal{D}_{H,\tau}.
\end{aligned}
$$

Since $\mathrm{trace}_{g_\theta}\tau = 0$, we deduce the first equality. The second equality follows directly from formula (10) and Proposition 3.4. □

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